

CPTH-9611477

q-alg/9611015

November 96

# A Generalization of $\mathcal{U}_h(sl(2))$ via Jacobian Elliptic Functions

A. Chakrabarti<sup>1</sup>

<sup>2</sup>*Centre de Physique Théorique, Ecole Polytechnique,  
91128 Palaiseau Cedex, France.*

## Abstract

A two-parametric generalization of the Jordanian deformation  $\mathcal{U}_h(sl(2))$  of  $sl(2)$  is presented. This involves Jacobian elliptic functions. In our deformation  $\mathcal{U}_{(h,k)}(sl(2))$ , for  $k^2 = 1$  one gets back  $\mathcal{U}_h(sl(2))$ . The construction is presented via a nonlinear map on  $sl(2)$ . This invertible map directly furnishes the highest weight irreducible representations of  $\mathcal{U}_{(h,k)}(sl(2))$ . This map also provides two distinct induced Hopf structures, which are exhibited. One is induced by the classical  $sl(2)$  and the other by the distinct one of  $\mathcal{U}_h(sl(2))$ . Automorphisms related to the two periods of the elliptic functions involved are constructed. Translations of one generator by half and quarter periods lead to interesting results in this context. Possibilities of applications are discussed briefly.

---

<sup>1</sup>chakra@orphee.polytechnique.fr

<sup>2</sup>Laboratoire Propre du CNRS UPR A.0014

# 1 Introduction

In [1] a nonlinear map relating the generators of the Jordanian  $\mathcal{U}_h(sl(2))$  and  $sl(2)$  was presented. Further applications of this mapping were studied in [2]. (A fairly complete list of references concerning  $\mathcal{U}_h(sl(2))$  can be found in [1] and [2].) Here a two-parametric generalization involving Jacobian elliptic functions of  $\mathcal{U}_h(sl(2))$  to  $\mathcal{U}_{(h,k)}(sl(2))$  is presented. For  $k^2 = 1$  one gets back  $\mathcal{U}_h(sl(2))$ . Setting now  $h = 0$  one gets  $sl(2)$ . The doubly periodic elliptic functions involved lead to new interesting features.

Though two distinct induced Hopf structures will be exhibited for  $\mathcal{U}_{(h,k)}(sl(2))$ , our main interest lies elsewhere. Exploring the enveloping algebra of  $sl(2)$  one encounters particularly interesting triplets leading to closed nonlinear algebras with remarkable properties. (Here the term nonlinear signifies that some of the commutators of the members of such a triplet give nonlinear functions of them.) One may try to explore higher dimensional algebras from this point of view. But so far our constructions have been limited to  $sl(2)$  and in that again mostly to two broad, complementary classes. In one [3,4] the nonlinearity arises through the generator  $J_0$  diagonalizable on the space of states spanning highest weight irreducible representations of the basic triplet  $(J_+, J_-, J_0)$ . In the other [1,2] this role is assumed by  $J_+$  (or alternatively by  $J_-$ ) which is nilpotent on such spaces. These two types of nonlinearity can be brought together. Such a construction will be presented elsewhere [5], combining the standard q-deformation  $\mathcal{U}_q(sl(2))$  with h-deformation leading to  $\mathcal{U}_{(q,h)}(sl(2))$ .

Here we stay within the second class in generalizing  $\mathcal{U}_h(sl(2))$ . The nonlinear map is first presented and then the Hopf structures and certain special automorphisms. A discussion of some interesting features and possibilities of applications will follow.

# 2 The construction as an invertible nonlinear map

Let  $(J_+, J_-, J_0)$  be the generators of  $sl(2)$  satisfying

$$[J_0, J_\pm] = \pm J_\pm \tag{1}$$

$$[J_+, J_-] = 2J_0 \tag{2}$$

Define the set  $(\hat{X}, \hat{Y}, J_0)$  through the Jacobian elliptic functions [6,7] and the parameters  $(h, k)$  as follows. Let

$$\frac{h}{2}J_+ = sn\left(\frac{h}{2}\hat{X}, k\right) \tag{3}$$

so that

$$\frac{h}{2}\hat{X} = sn^{-1}\left(\frac{h}{2}J_+, k\right). \tag{4}$$

Let

$$\hat{Y} = g(J_+)J_-g(J_+) \tag{5}$$

where

$$g(J_+) = \left( (1 - (\frac{h}{2}J_+)^2)(1 - k^2(\frac{h}{2}J_+)^2) \right)^{\frac{1}{4}} \quad (6)$$

$$= \left( cn(\frac{h}{2}\hat{X}, k) dn(\frac{h}{2}\hat{X}, k) \right)^{\frac{1}{2}} \quad (7)$$

$$\equiv g(X) \quad (8)$$

so that

$$J_- = g^{-1}(\hat{X})\hat{Y}g^{-1}(\hat{X}). \quad (9)$$

This map is to be understood here (at least to start with) in the context of the standard highest weight representations of  $sl(2)$  of dimension  $(2j+1)$  for  $j$  (half)integer. Since  $J_+$  is nilpotent on such spaces the series developments of the elliptic functions involved are truncated. No convergence problems arise. Thus, for example, from (3) and (4) setting  $j = \frac{5}{2}$  keeping terms upto the fifth order only

$$\frac{h}{2}J_+ = \frac{h}{2}\hat{X} - \frac{1}{3!}(1+k^2)(\frac{h}{2}\hat{X})^3 + \frac{1}{5!}(1+14k+k^4)(\frac{h}{2}\hat{X})^5 \quad (10)$$

$$\frac{h}{2}\hat{X} = \frac{h}{2}J_+ + \frac{1}{3!}(1+k^2)(\frac{h}{2}J_+)^3 + \frac{1}{5!}(9+6k^2+9k^4)(\frac{h}{2}J_+)^5 \quad (11)$$

The series for other elliptic functions and their inverses are also truncated analogously. From (1) to (9) one obtains for the triplet  $(\hat{X}, \hat{Y}, J_0)$  the commutators

$$[\hat{X}, \hat{Y}] = 2J_0 \quad (12)$$

$$[J_0, \hat{X}] = \frac{2}{h} \left( \frac{sn(\frac{h}{2}\hat{X}, k)}{cn(\frac{h}{2}\hat{X}, k)dn(\frac{h}{2}\hat{X}, k)} \right) \equiv G(\hat{X}) \quad (13)$$

$$[J_0, \hat{Y}] = -\frac{1}{2}(f(\hat{X})\hat{Y} + \hat{Y}f(\hat{X})) \quad (14)$$

$$f(\hat{X}) = \frac{1 - k^2(sn(\frac{h}{2}\hat{X}, k))^4}{(cn(\frac{h}{2}\hat{X}, k)dn(\frac{h}{2}\hat{X}, k))^2} \quad (15)$$

$$= \frac{2}{sn(h\hat{X}, k)} \left( \frac{sn(\frac{h}{2}\hat{X}, k)}{cn(\frac{h}{2}\hat{X}, k)dn(\frac{h}{2}\hat{X}, k)} \right) \quad (16)$$

$$= \frac{1 - k^2(\frac{h}{2}J_+)^4}{(1 - (\frac{h}{2}J_+)^2)(1 - k^2(\frac{h}{2}J_+)^2)} \quad (17)$$

For  $k^2 = 1$ , denoting

$$(\hat{X})_{k^2=1} = X \quad (18)$$

$$(\hat{Y})_{k^2=1} = Y \quad (19)$$

$$\frac{h}{2}X = \operatorname{arctanh}\left(\frac{h}{2}J_+\right) \quad (20)$$

$$Y = \left(1 - \left(\frac{h}{2}J_+\right)^2\right)^{\frac{1}{2}} J_- \left(1 - \left(\frac{h}{2}J_+\right)^2\right)^{\frac{1}{2}} \quad (21)$$

This is the nonlinear map introduced in [1] to obtain  $\mathcal{U}_h(sl(2))$  satisfying

$$[X, Y] = 2J_0 \quad (22)$$

$$[J_0, X] = \frac{1}{h} \sinh(hX) \quad (23)$$

$$[J_0, Y] = -\frac{1}{2} \left( \cosh(hX)Y + Y \cosh(hX) \right) \quad (24)$$

One can now express  $(\hat{X}, \hat{Y})$  also through the map

$$\frac{h}{2}\hat{X} = sn^{-1}(\tanh(\frac{h}{2}X), k) \quad (25)$$

$$\hat{Y} = \left( \frac{1 - k^2(\tanh(\frac{h}{2}X))^2}{1 - (\tanh(\frac{h}{2}X))^2} \right)^{\frac{1}{4}} Y \left( \frac{1 - k^2(\tanh(\frac{h}{2}X))^2}{1 - (\tanh(\frac{h}{2}X))^2} \right)^{\frac{1}{4}} \quad (26)$$

The unique Casimir for  $sl(2)$  can be expressed in terms of the different triplets as follows

$$C = J_- J_+ + J_0(J_0 + 1) \quad (27)$$

$$= \frac{2}{h} \cosh(\frac{h}{2}X) Y \sinh(\frac{h}{2}X) + J_0(J_0 + 1) \quad (28)$$

$$= \frac{2}{h} (cn(\frac{h}{2}\hat{X}, k) dn(\frac{h}{2}\hat{X}, k))^{-\frac{1}{2}} \hat{Y} (cn(\frac{h}{2}\hat{X}, k) dn(\frac{h}{2}\hat{X}, k))^{-\frac{1}{2}} sn(\frac{h}{2}\hat{X}, k) + J_0(J_0 + 1) \quad (29)$$

The matrix elements of  $\hat{X}, \hat{Y}$  on the standard basis states  $|j, m\rangle$  of  $sl(2)$  are obtained as follows. One has

$$J_+ |j, m\rangle = a_m |j, m+1\rangle \quad (30)$$

$$J_- |j, m\rangle = a_{m-1} |j, m-1\rangle \quad (31)$$

$$J_0 |j, m\rangle = m |j, m\rangle \quad (32)$$

$$a_m = ((j-m)(j+m+1))^{\frac{1}{2}}, \quad (m = -j, \dots, j) \quad (33)$$

Using the series developments [6,7] for the elliptic functions truncated by

$$J_+^{(2j+1)} |j, m\rangle = 0 \quad (34)$$

one obtains

$$\begin{aligned} \hat{X} |j, m\rangle &= a_m |j, m+1\rangle + \left(\frac{h}{2}\right)^2 \frac{1}{3!} (1+k^2) \left(\prod_{i=0}^2 a_{m+i}\right) |j, m+3\rangle \\ &+ \left(\frac{h}{2}\right)^4 \frac{1}{5!} (1+14k^2+k^4) \left(\prod_{i=0}^4 a_{m+i}\right) |j, m+5\rangle + \dots \end{aligned} \quad (35)$$

Similarly one can evaluate

$$\begin{aligned} \hat{Y} | j, m \rangle = & \left\{ \left( 1 - \frac{1}{4}(1+k^2) \left( \frac{h}{2} J_+ \right)^2 - \frac{1}{32}(3+k^2) \left( \frac{h}{2} J_+ \right)^4 \dots \right) J_- \right. \\ & \left. \left( 1 - \frac{1}{4}(1+k^2) \left( \frac{h}{2} J_+ \right)^2 - \frac{1}{32}(3+k^2) \left( \frac{h}{2} J_+ \right)^4 \dots \right) \right\} | j, m \rangle \end{aligned} \quad (36)$$

For  $j = \frac{5}{2}$  for example, the terms exhibited explicitly in the series suffice. One has for each triplet

$$C | j, m \rangle = j(j+1) | j, m \rangle. \quad (37)$$

It can be shown that (12),(13) and (14) are compatible with the Jacobi identity. The crucial relation that assures this is  $f(\hat{X}) = \frac{d}{d\hat{X}} G(\hat{X})$ .

### 3 The two induced Hopf structures

Let us now indicate the induced Hopf structures corresponding to the maps (4), (5) and (25), (26) respectively. It is sufficient to consider the coproducts as illustrations. The counits and the antipodes can be treated analogously.

For  $sl(2)$  one has

$$\Delta J_i = J_i \otimes 1 + 1 \otimes J_i, \quad (i = \pm, 0) \quad (38)$$

These induce, through (4), (5) and (6), the first structure. Namely,

$$\Delta_1 \hat{X} = \frac{2}{h} s n^{-1} \left( \frac{h}{2} \Delta J_+, k \right) \quad (39)$$

$$\Delta_1 \hat{Y} = g(\Delta J_+) (\Delta J_-) g(\Delta J_+) \quad (40)$$

$$\Delta J_0 = J_0 \otimes 1 + 1 \otimes J_0 \quad (41)$$

Inverting the map (4), (5) one can express the righthand sides in terms of  $(\hat{X}, \hat{Y}, J_0)$ . Setting  $k^2 = 1$  one obtains the induced coproducts  $(\Delta_1)$  of  $\mathcal{U}_h(sl(2))$ .

But as is wellknown,  $\mathcal{U}_h(sl(2))$  has a distinct Hopf structure leading to a nontrivial  $R$ -matrix. ( See [1] and the references cited there.) The coproducts corresponding to this one are

$$\Delta X = X \otimes 1 + 1 \otimes X \quad (42)$$

$$\Delta Y = Y \otimes e^{hX} + e^{-hX} \otimes Y \quad (43)$$

$$\Delta J_0 = J_0 \otimes e^{hX} + e^{-hX} \otimes J_0 \quad (44)$$

Through (25), (26) one thus has a second set of induced coproducts

$$\Delta_2 \hat{X} = \frac{2}{h} sn^{-1}(\tanh(\frac{h}{2} \Delta X), k) \quad (45)$$

$$\Delta_2 \hat{Y} = \left( \frac{1 - k^2 (\tanh(\frac{h}{2} \Delta X))^2}{1 - (\tanh(\frac{h}{2} \Delta X))^2} \right)^{\frac{1}{4}} \Delta Y \left( \frac{1 - k^2 (\tanh(\frac{h}{2} \Delta X))^2}{1 - (\tanh(\frac{h}{2} \Delta X))^2} \right)^{\frac{1}{4}} \quad (46)$$

$$\Delta_2 J_0 = J_0 \otimes e^{hX} + e^{-hX} \otimes J_0 \quad (47)$$

Inverting the map (25), (26) the righthand sides can be expressed again in terms of  $(\hat{X}, \hat{Y}, J_0)$ . Thus, for example

$$\Delta_1 \hat{X} = \frac{2}{h} sn^{-1} \left( sn(\frac{h}{2} \hat{X}, k) \otimes 1 + 1 \otimes sn(\frac{h}{2} \hat{X}, k), k \right) \quad (48)$$

$$\Delta_2 \hat{X} = \frac{2}{h} sn^{-1} \left( \tanh(\tanh^{-1} sn(\frac{h}{2} \hat{X}, k) \otimes 1 + 1 \otimes \tanh^{-1} sn(\frac{h}{2} \hat{X}, k)), k \right) \quad (49)$$

One can similarly reexpress  $\Delta_1 \hat{Y}$  and  $\Delta_2 \hat{Y}$ . For the non cocommutative  $\Delta_2$  the R-matrices of  $\mathcal{U}_h(sl(2))$  (of which some examples can be found in [1] ) become relevant. But we will not persue further these aspects in this paper.

We have considered only induced Hopf structures. No new distinct one has been obtained.

## 4 Automorphisms corresponding to half and quarter periods of elliptic functions

For comparison, we start by recapitulating the  $\mathcal{U}_h(sl(2))$  automorphisms studied in [1]. Corresponding to the standard  $sl(2)$  involution

$$(J_+, J_-, J_0) \rightarrow (-J_+, -J_-, J_0) \quad (50)$$

$$(X, Y, J_0) \rightarrow (-X, -Y, J_0). \quad (51)$$

Here the limit  $h \rightarrow 0$  is straightforward. The situation is similar for  $(\hat{X}, \hat{Y}, J_0)$  corresponding to (50).

But corresponding to half period of tanh one also has the automorphism

$$(X, Y, J_0) \rightarrow (X + \frac{i\pi}{h}, -Y, -J_0) \quad (52)$$

which can be iterated. This has no straightforward  $h \rightarrow 0$  (classical) limit. Since

$$\tanh(\frac{h}{2}(X + \frac{i\pi}{h})) = \coth(\frac{h}{2}X) \quad (53)$$

formally, from (20), (21) and (53)

$$(J_+, J_-, J_0) \rightarrow (J'_+, J'_-, J'_0) \quad (54)$$

$$J'_+ = \left(\frac{2}{h}\right)^2 J_+^{-1} \quad (55)$$

$$J'_- = \left(\frac{h}{2}\right)^2 J_+ J_- J_+ \quad (56)$$

$$J'_0 = -J_0 \quad (57)$$

This is an involution consistently with the fact that an iteration of (52)

$$(X, Y, J_0) \rightarrow \left(X + \frac{i2\pi}{h}, Y, J_0\right) \quad (58)$$

is trivial at the level of  $(J_+, J_-, J_0)$ .

The action of  $J'_+$  on the  $|j, m\rangle$  basis (see (30) to (33)) is no longer well defined since  $a_m^{-1}$  diverges for  $m = j$ . But the nonlinear map assures that for  $X$  there is only an additive imaginary term with

$$X |j, j\rangle = \frac{i\pi}{h} |j, j\rangle \quad (59)$$

This "half period automorphism" leads to a special class of  $R$ -matrices for  $\mathcal{U}_h(sl(2))$  [1]. Let us now construct its generalization to the elliptic case.

The two periods of the elliptic functions [6,7] are given in, terms of  $K$  and  $K'$  where

$$K = F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} (I - k^2 \sin^2 t)^{-\frac{1}{2}} dt \quad (60)$$

$$K' = F\left(\frac{\pi}{2}, (1 - k^2)^{\frac{1}{2}}\right) \quad (61)$$

The primitive periods for  $sn, cn, dn$  are respectively

$$(4K, 2iK'), \quad (4K, 2(K + iK')), \quad (2K, 4iK') \quad (62)$$

Translations of  $(\frac{h}{2}\hat{X})$  by an entire common period leaves the elliptic functions unchanged and hence the algebra is left evidently invariant. But from the results (Table 7, page 350 [6]) corresponding to half and quarter periods one can construct also the following automorphisms

$$(\hat{X}, \hat{Y}, J_0) \rightarrow \left(\hat{X} + \frac{2}{h}iK', -\hat{Y}, -J_0\right) \quad (63)$$

$$(\hat{X}, \hat{Y}, J_0) \rightarrow \left(\hat{X} + \frac{2}{h}(2K + iK'), -\hat{Y}, -J_0\right) \quad (64)$$

Note that the imaginary translation  $iK'$  of the argument  $(\frac{h}{2}\hat{X})$  corresponds to a quarter period for  $dn$ . But the combined effect of all the factors leads to the following involutions

(to be compared with (54)). For nonzero values of  $(h, k)$  one has with  $\epsilon = \pm 1$  for (63), (64) respectively

$$(J_+, J_-, J_0) \rightarrow (\epsilon \frac{1}{k} (\frac{2}{h})^2 J_+^{-1}, \epsilon k (\frac{h}{2})^2 J_+ J_- J_+, -J_0). \quad (65)$$

Some relevant comments are added in the concluding discussions.

## 5 Discussion

We have presented our formalism, namely, the nonlinear map, the representations and the automorphisms. Further developments can now be envisaged. The applications of  $h$ -deformation to  $so(4)$  and  $e(3)$  have been studied in [2]. One can try to generalize them in the context of  $(h, k)$  deformation presented here.

One can try to select suitable classes of ansatz for Hamiltonians  $H(\hat{X}, \hat{Y}, J_0)$  expressed directly as functions of these operators and explore their properties. Our nonlinear map will provide a powerful tool in the study of the spectra of such Hamiltonians.

In the context of magnetic fields perpendicular to a plane  $\mathcal{U}_q(sl(2))$  has been used to construct Hamiltonians [8]. A classical limit has also been studied [9]. In [10] a vector potential given by the Weierstrass zeta function has been related to infinite dimensional representations of  $\mathcal{U}_q(sl(2))$ .

We have constructed two parametric deformations, here for  $sl(2)$  and in [5] in the context of  $\mathcal{U}_q(sl(2))$ . In our constructions the second parameter also plays a full-fledged role. It appears in the algebra and is not relegated to the coproducts only. One can try to explore, not necessarily for magnetic fields only, but in a broader context of models the potential role of these parameters.

Let us finally come back to our automorphisms. In [11] outer automorphisms of infinite dimensional representations of  $sl(2)$  involving  $J_+^{-1}$  ( $s_+^{-1}$  in the notation of [11]) have been used to construct dynamical  $r$ -matrices. In (65) we have exhibited involutions leading to the inversion  $J_+^{-1}$ . At the level of the basic triplet  $(J_+, J_-, J_0)$  such inversions permit only infinite dimensional (no highest weight) representations. But for the triplet  $(\hat{X}, \hat{Y}, J_0)$  or  $(X, Y, J_0)$  they still permit finite dimensional (though complex) representations. The explanation, here, is simple enough. (When the eigenvalue of  $X$ , say, on the highest weight state  $|j, j\rangle$  is  $(\frac{i\pi}{h})$  that of  $\tanh(\frac{h}{2}X)$  diverges.) But the possibility and the consequences of having finite dimensional representations for suitably selected triplets in the enveloping algebra, when the realization of the basic generators themselves permit only infinite dimensional ones, are worth exploring more generally and systematically. Complementarily, going beyond the scope of this paper one can envisage independent and intrinsic explorations of infinite dimensional representations for the nonlinear algebras generated by  $(\hat{X}, \hat{Y}, J_0)$  or  $(X, Y, J_0)$ .

Elliptic functions appeared in deformations of  $sl(2)$  in a quite different fashion and with an extra generator in the works of Sklyanin [12]. References to recent "elliptic" deformations can be found in [4].



It is a pleasure to thank Boucif Abdesselam for discussions and varied help. Our work related to  $h$ -deformations began in collaboration with and due to the initiative of R. Chakrabarti.

## Citations

- [1] B. Abdesselam, A. Chakrabarti and R. Chakrabarti, *Irreducible representations of the Jordanian quantum algebra via a nonlinear map*, Ecole Polytechnique Preprint CPTH-S455.0696.( To be published in Mod.Phys. Lett.A.)
- [2] B. Abdesselam, A. Chakrabarti and R. Chakrabarti, *On  $\mathcal{U}_h(sl(2))$ ,  $\mathcal{U}_h(e_3)$  and their representations*, Ecole Polytechnique Preprint CPTH-S458.0696. (To be published in Jour. Mod. Phys. A.)
- [3] B. Abdesselam, J. Beckers, A. Chakrabarti and N. Debergh, J. Phys. A: Math. Gen. **29**, 3075 (1996).
- [4] A. Chakrabarti, *Induced Hopf structure and irreducible representations of an elliptic  $\mathcal{U}_{q,p}(sl(2))$  via a nonlinear map*, Ecole Polytechnique Preprint CPTH-9611476.
- [5] A. Chakrabarti and R. Chakrabarti, *Combined  $(q, h)$  deformation as a nonlinear map on  $\mathcal{U}_q(sl(2))$* , Ecole Polytechnique Preprint (in preparation).
- [6] A. Erdelyi et al. *Higher Transcendental Functions*, Vol 2., Ch.13, McGraw Hill N.Y. (1953)
- [7] I.S. Gradshteyn and Ryzhik, *Table of Integrals, Series and Products*, Sec. 8.1 ; Academic Press (1965)
- [8] P.B. Wiegmann and A.V. Zabrodin, Nucl. Phys. B **422**, 495 (1994)
- [9] E.G. Floratos and S. Nicolis, *An  $su(2)$  analog of the Azbel-Hopfstadler Hamiltonian* (LPTENS 95/37; hep-th/9508111)
- [10] A. Arai, Jour. Math. Phys. **37**, 4203 (1996)
- [11] A.V. Tsiganov, *Automorphisms of  $su(2)$  and dynamical  $r$ -matrices* (St. Petersburg Univ. preprint, 1996; solv-int/9610003)
- [12] E.K. Sklyanin , Funct. Anal. Appl. **16**, 263 (1982); **17**, 273 (1983)